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A vertex operator approach for correlation functions of Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model

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Abstract
Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model is considered on the basis of bosonization of vertex operators in the $A^{(1)}_{n-1}$ model and vertex–face transformation. The corner transfer matrix (CTM) Hamiltonian of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model and tail operators are expressed in terms of bosonized vertex operators in the $A^{(1)}_{n-1}$ model. Correlation functions of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model can be obtained by using these objects, in principle. In particular, we calculate spontaneous polarization, which reproduces the result we obtained in 1993.

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Dedicated to Professor Tetsuji Miwa on the occasion of his 60th birthday

1. Introduction

In this paper we consider Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model [1] on the basis of bosonization of vertex operators in the $A^{(1)}_{n-1}$ model [2] and vertex–face transformation. Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model is a higher rank generalization of Baxter's eight-vertex model [3] in the sense that the former model is an n-state model. The $(\mathbb{Z}/n\mathbb{Z})$-symmetric model is a vertex model on a two-dimensional lattice such that the state variables take on values of $(\mathbb{Z}/n\mathbb{Z})$-spin. A local weight R^{ik}_{jl} is assigned to spin configuration j, l, i, k around a vertex. The model is $(\mathbb{Z}/n\mathbb{Z})$-symmetric in a sense that R^{ik}_{jl} satisfies the two conditions: (i) $R^{ik}_{jl} = 0$ unless $j+l = i+k$ (mod n) and (ii) $R^{i+pk+p}_{j+pl+p} = R^{ik}_{jl}$ for any $p \in (\mathbb{Z}/n\mathbb{Z})$. Since there are n^3 non-zero weights among R^{ik}_{jl}'s, we may call the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model by the n^3-vertex model. (When $n = 2$, it becomes the eight-vertex model.)

In [4], Lashkevich and Pugai presented the integral formulae for correlation functions of the eight-vertex model [3] using bosonization of vertex operators in the eight-vertex SOS model [5] and vertex–face transformation. The present paper aims to give an $sl(n)$-generalization of Lashkevich–Pugai's construction. For our purpose, we use the vertex–face correspondence between the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model and unrestricted $A^{(1)}_{n-1}$ model. First, we note that the $A^{(1)}_{n-1}$ model [6] is a restricted model, while we should relate the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model to the unrestricted $A^{(1)}_{n-1}$ model. Second, we note that the original vertex–face correspondence [6] maps the $A^{(1)}_{n-1}$ model in regime III to the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model in the disordered phase. We should relate the former to the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model in the principal regime.

In this paper, we present integral formulae for correlation functions of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model on the basis of the free field representation formalism. As the simplest example, we perform the calculation of the integral formulae for a one-point function, in order to obtain the spontaneous polarization of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model.

There is another approach to find the expression for correlation functions. It was shown in [7] that the correlation functions of the eight-vertex model satisfy a set of difference equations, the quantum Knizhnik–Zamolodchikov equation of level -4. On the basis of the difference equation approach, we obtained the expression of the spontaneous polarization of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model [8]. In this paper, we show that the expressions for the spontaneous polarization of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model obtained on the basis of the free field representation formalism reproduce the known result in [8]. This coincidence indicates the relevance of the free field representation formalism.

The present paper is organized as follows. In section 2, we review the basic definitions of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model [1], the corresponding dual face model [6] and the vertex–face correspondence. In section 3, we introduce the corner transfer matrix (CTM) Hamiltonians and the vertex operators of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model and $A^{(1)}_{n-1}$ model, and also introduce the tail operators which relates those two CTM Hamiltonians. In section 4 we construct the free field formalism of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model. In section 5 we present trace formulae for correlation functions of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model. Furthermore, we calculate the spontaneous polarization of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model in this formalism. Sections 4 and 5 are main original parts of the present paper. In section 6 we give some concluding remarks.

2. Basic definitions

The present section aims to formulate the problem, thereby fixing the notation.

2.1. Theta functions

The Jacobi theta function with two pseudo-periods 1 and τ ($\mathrm{Im}\tau > 0$) are defined as follows:

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(v;\tau) := \sum_{m\in\mathbb{Z}} \exp\{\pi\sqrt{-1}(m+a)[(m+a)\tau + 2(v+b)]\}, \tag{2.1}$$

for $a, b \in \mathbb{R}$. Let $n \in \mathbb{Z}_{\geqslant 2}$ and $r \in \mathbb{R}$ such that $r > n-1$, and also fix the parameter x such that $0 < x < 1$. We will use the abbreviations,

$$[v] = x^{\frac{v^2}{r}-v}\Theta_{x^{2r}}(x^{2v}), \qquad [v]' = x^{\frac{v^2}{r-1}-v}\Theta_{x^{2r-2}}(x^{2v}), \tag{2.2}$$

where

$$\Theta_q(z) = (z;q)_\infty (qz^{-1};q)_\infty (q;q)_\infty = \sum_{m\in\mathbb{Z}} q^{m(m-1)/2}(-z)^m,$$

$$(z;q_1,\ldots,q_m) = \prod_{i_1,\ldots,i_m\geqslant 0} \left(1 - zq_1^{i_1}\ldots q_m^{i_m}\right).$$

Note that

$$\vartheta\begin{bmatrix}1/2\\-1/2\end{bmatrix}\left(\frac{v}{r},\frac{\pi\sqrt{-1}}{\epsilon r}\right)=\sqrt{\frac{\epsilon r}{\pi}}\exp\left(-\frac{\epsilon r}{4}\right)[v],$$

where $x = e^{-\epsilon}$ $(\epsilon > 0)$.

For later conveniences we also introduce the following symbols:

$$r_l(v) = z^{\frac{r-1}{r}\frac{n-l}{n}}\frac{g_l(z^{-1})}{g_l(z)}, \qquad g_l(z) = \frac{\{x^{2n+2r-l-1}z\}\{x^{l+1}z\}}{\{x^{2n-l+1}z\}\{x^{2r+l-1}z\}}, \tag{2.3}$$

where $z = x^{2v}$, $1 \leqslant l \leqslant n$ and

$$\{z\} = (z; x^{2r}, x^{2n})_\infty. \tag{2.4}$$

These factors will appear in the commutation relations among the type I vertex operators.

The integral kernel for the type I vertex operators will be given as the products of the following elliptic functions:

$$f(v,w) = \frac{\left[v+\frac{1}{2}-w\right]}{\left[v-\frac{1}{2}\right]}, \qquad g(v) = \frac{[v-1]}{[v+1]}. \tag{2.5}$$

2.2. Belavin's vertex model

Let $V = \mathbb{C}^n$ and $\{\varepsilon_\mu\}_{0\leqslant\mu\leqslant n-1}$ be the standard orthonormal basis with the inner product $\langle\varepsilon_\mu, \varepsilon_\nu\rangle = \delta_{\mu\nu}$. Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model is a vertex model on a two-dimensional square lattice $\mathcal{L}$ such that the state variables take on values of $(\mathbb{Z}/n\mathbb{Z})$-spin. In the original papers [1, 9], the R-matrix in the disordered phase is given. For the present purpose, we need the following R-matrix:

$$R(v) = \frac{[1]}{[1-v]}r_1(v)\overline{R}(v), \quad \overline{R}(v) = \frac{1}{n}\sum_{\alpha\in G_n}\frac{\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha_1}{n}\\ \frac{1}{2}+\frac{\alpha_2}{n}\end{bmatrix}\left(\frac{1}{nr}-\frac{v}{r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}{\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha_1}{n}\\ \frac{1}{2}+\frac{\alpha_2}{n}\end{bmatrix}\left(\frac{1}{nr};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}I_\alpha\otimes I_\alpha^{-1}. \tag{2.6}$$

Here $G_n = (\mathbb{Z}/n\mathbb{Z})\times(\mathbb{Z}/n\mathbb{Z})$, and $I_\alpha = g^{\alpha_1}h^{\alpha_2}$ for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, where

$$gv_i = \omega^i v_i, \qquad hv_i = v_{i-1}, \tag{2.7}$$

with $\omega = \exp(2\pi\sqrt{-1}/n)$. We assume that the parameters v, ϵ and r lie in the so-called principal regime:

$$\epsilon > 0, \qquad r > 1, \qquad 0 < v < 1. \tag{2.8}$$

When $n = 2$, the principal regime (2.8) lies in one of the antiferroelectric phases of the eight-vertex model [3]. We describe n kinds of ground states of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model in the principal regime in section 3.1.

The R-matrix satisfies the Yang–Baxter equation (YBE),

$$R_{12}(v_1-v_2)R_{13}(v_1-v_3)R_{23}(v_2-v_3) = R_{23}(v_2-v_3)R_{13}(v_1-v_3)R_{12}(v_1-v_2), \tag{2.9}$$

where $R_{ij}(v)$ denotes the matrix on $V^{\otimes 3}$, which acts as $R(v)$ on the ith and jth components and as identity on the other one.

If $i+k=j+l$ (mod n), the elements of the R-matrix $\overline{R}(v)_{jl}^{ik}$ is given as follows:

$$\overline{R}(v)_{jl}^{ik}=\frac{h(v)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\frac{k-i}{n}\end{bmatrix}\left(\frac{1-v}{nr};\frac{\pi\sqrt{-1}}{n\epsilon r}\right)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\frac{j-k}{n}\end{bmatrix}\left(\frac{v}{nr};\frac{\pi\sqrt{-1}}{n\epsilon r}\right)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\frac{j-i}{n}\end{bmatrix}\left(\frac{1}{nr};\frac{\pi\sqrt{-1}}{n\epsilon r}\right)},\tag{2.10}$$

where

$$h(v)=\prod_{j=0}^{n-1}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\frac{j}{n}\end{bmatrix}\left(\frac{v}{nr};\frac{\pi\sqrt{-1}}{n\epsilon r}\right)\Bigg/\prod_{j=1}^{n-1}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\frac{j}{n}\end{bmatrix}\left(0;\frac{\pi\sqrt{-1}}{n\epsilon r}\right),$$

and otherwise $\overline{R}(v)_{jl}^{ik}=0$.

Note that the weights (2.10) reproduce those of the eight-vertex model in the principal regime when $n=2$ [3].

2.3. The weight lattice and the root lattice of $A_{n-1}^{(1)}$

Let $V=\mathbb{C}^n$ and $\{\varepsilon_\mu\}_{0\leqslant\mu\leqslant n-1}$ be the standard orthonormal basis as before. The weight lattice of $A_{n-1}^{(1)}$ is defined as follows:

$$P=\bigoplus_{\mu=0}^{n-1}\mathbb{Z}\bar{\varepsilon}_\mu,\tag{2.11}$$

where

$$\bar{\varepsilon}_\mu=\varepsilon_\mu-\varepsilon,\qquad \varepsilon=\frac{1}{n}\sum_{\mu=0}^{n-1}\varepsilon_\mu.$$

We denote the fundamental weights by $\omega_\mu(1\leqslant\mu\leqslant n-1)$,

$$\omega_\mu=\sum_{\nu=0}^{\mu-1}\bar{\varepsilon}_\nu,$$

and also denote the simple roots by $\alpha_\mu(1\leqslant\mu\leqslant n-1)$,

$$\alpha_\mu=\varepsilon_{\mu-1}-\varepsilon_\mu=\bar{\varepsilon}_{\mu-1}-\bar{\varepsilon}_\mu.$$

The root lattice of $A_{n-1}^{(1)}$ is defined as follows:

$$Q=\bigoplus_{\mu=1}^{n-1}\mathbb{Z}\alpha_\mu,\tag{2.12}$$

For $a\in P$ we set

$$a_{\mu\nu}=\bar{a}_\mu-\bar{a}_\nu,\qquad \bar{a}_\mu=\langle a+\rho,\varepsilon_\mu\rangle=\langle a+\rho,\bar{\varepsilon}_\mu\rangle,\qquad \rho=\sum_{\mu=1}^{n-1}\omega_\mu.\tag{2.13}$$

Useful formulae are

$$\langle\bar{\varepsilon}_\mu,\varepsilon_\nu\rangle=\langle\bar{\varepsilon}_\mu,\bar{\varepsilon}_\nu\rangle=\delta_{\mu\nu}-\frac{1}{n},\qquad \langle\alpha_\mu,\omega_\nu\rangle=\delta_{\mu\nu},$$

$$\langle\bar{\varepsilon}_\mu,\omega_\nu\rangle=\theta(\mu<\nu)-\frac{\nu}{n},\qquad \langle\omega_\mu,\omega_\nu\rangle=\min(\mu,\nu)-\frac{\mu\nu}{n}.$$

When $a+\rho=\sum_{\mu=0}^{n-1}k^\mu\omega_\mu$, we have $a_{\mu\nu}=k^{\mu+1}+\cdots+k^\nu$ when $\mu<\nu$, and

$$\langle a+\rho, a+\rho\rangle=\frac{1}{n}\sum_{\mu<\nu}a_{\mu\nu}^2, \qquad \langle a+\rho,\rho\rangle=\frac{1}{2}\sum_{\mu<\nu}a_{\mu\nu}.$$

Let $\sum_{\mu=0}^{n-1}k^\mu=r$, where $a+\rho=\sum_{\mu=0}^{n-1}k^\mu\omega_\mu$, then we denote $a\in P_{r-n}$.

2.4. The $A_{n-1}^{(1)}$ face model

An ordered pair $(a,b)\in P_{r-n}^2$ is called *admissible* if $b=a+\bar{\varepsilon}_\mu$, for a certain $\mu (0\leqslant\mu\leqslant n-1)$. For $(a,b,c,d)\in P_{r-n}^4$, let $W\left[\begin{smallmatrix}c & d\\ b & a\end{smallmatrix}\middle| v\right]$ be the Boltzmann weight of the $A_{n-1}^{(1)}$ model for the state configuration $\left[\begin{smallmatrix}c & d\\ b & a\end{smallmatrix}\right]$ round a face. Here the four states a, b, c and d are ordered clockwise from the SE corner. In this model, $W\left[\begin{smallmatrix}c & d\\ b & a\end{smallmatrix}\middle| v\right]=0$ unless the four pairs (a,b), (a,d), (b,c) and (d,c) are admissible. Non-zero Boltzmann weights are parametrized in terms of the elliptic theta function of the spectral parameter v as follows:

$$W\left[\begin{matrix}a+2\bar{\varepsilon}_\mu & a+\bar{\varepsilon}_\mu\\ a+\bar{\varepsilon}_\mu & a\end{matrix}\middle| v\right]=r_1(v),$$

$$W\left[\begin{matrix}a+\bar{\varepsilon}_\mu+\bar{\varepsilon}_\nu & a+\bar{\varepsilon}_\mu\\ a+\bar{\varepsilon}_\nu & a\end{matrix}\middle| v\right]=-r_1(v)\frac{[v][a_{\mu\nu}+1]}{[1-v][a_{\mu\nu}]} \qquad (\mu\neq\nu), \tag{2.14}$$

$$W\left[\begin{matrix}a+\bar{\varepsilon}_\mu+\bar{\varepsilon}_\nu & a+\bar{\varepsilon}_\mu\\ a+\bar{\varepsilon}_\mu & a\end{matrix}\middle| v\right]=r_1(v)\frac{[1][v+a_{\mu\nu}]}{[1-v][a_{\mu\nu}]} \qquad (\mu\neq\nu).$$

We consider the so-called regime III in the model, i.e., $0<v<1$.

The Boltzmann weights (2.14) solve the YBE for the face model [6]:

$$\begin{aligned}\sum_g W\left[\begin{matrix}d & e\\ c & g\end{matrix}\middle| v_1\right]W\left[\begin{matrix}c & g\\ b & a\end{matrix}\middle| v_2\right]W\left[\begin{matrix}e & f\\ g & a\end{matrix}\middle| v_1-v_2\right]\\ =\sum_g W\left[\begin{matrix}g & f\\ b & a\end{matrix}\middle| v_1\right]W\left[\begin{matrix}d & e\\ g & f\end{matrix}\middle| v_2\right]W\left[\begin{matrix}d & g\\ c & b\end{matrix}\middle| v_1-v_2\right].\end{aligned} \tag{2.15}$$

2.5. Vertex–face correspondence

In this paper, we use the R-matrix of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model in the principal regime while Belavin's original paper used that in the disordered phase. Thus, we need different intertwining vectors from that by Jimbo–Miwa–Okado [6].

Let

$$t(v)_{a-\bar{\varepsilon}_\mu}^a=\sum_{\nu=0}^{n-1}\varepsilon_\nu\vartheta\left[\begin{matrix}0\\ \frac{1}{2}+\frac{\nu}{n}\end{matrix}\right]\left(\frac{v}{nr}+\frac{\bar{a}_\mu}{r};\frac{\pi\sqrt{-1}}{n\epsilon r}\right). \tag{2.16}$$

Then we have (cf. figure 1)

$$R(v_1-v_2)t(v_1)_a^d\otimes t(v_2)_d^c=\sum_b t(v_1)_b^c\otimes t(v_2)_a^b W\left[\begin{matrix}c & d\\ b & a\end{matrix}\middle| v_1-v_2\right]. \tag{2.17}$$

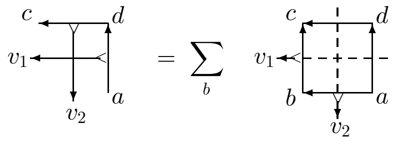


Figure 1. Picture representation of the vertex–face correspondence.

3. Vertex–face transformation

The basic objects in the vertex operator approach are the CTMs and the vertex operators [10]. In sections 3.1 and 3.2 we recall the CTM Hamiltonians, the type I vertex operators and the space of states of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model and the $A^{(1)}_{n-1}$ model, respectively.

In [4], Lashkevich and Pugai introduced the nonlocal operator called the tail operator, in order to express the correlation functions of the eight-vertex model in terms of those of the SOS model. In section 3.3, we introduce the tail operator for the present purpose; i.e., in order to express the correlation functions of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model in terms of those of the $A^{(1)}_{n-1}$ model. The commutation relations among the tail operators and the type I vertex operators are given in section 3.4.

3.1. The CTM Hamiltonian for the vertex model

Let us consider the 'low-temperature' limit $x \to 0$. Then the elements of the R-matrix behave as

$$R^{\mu\nu}_{\mu'\nu'}(v) \sim \zeta^{H_v(\mu,\nu)}\delta^{\mu}_{\nu'}\delta^{\nu}_{\mu'}, \tag{3.1}$$

where $z = x^{2v} = \zeta^n$ and

$$H_v(\mu,\nu) = \begin{cases} \mu - \nu - 1 & \text{if} \quad 0 \leqslant \nu < \mu \leqslant n-1 \\ n - 1 + \mu - \nu & \text{if} \quad 0 \leqslant \mu \leqslant \nu \leqslant n-1. \end{cases} \tag{3.2}$$

Thus the CTM Hamiltonian of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model in the principal regime is given as follows:

$$H_{\mathrm{CTM}}(\mu_1,\mu_2,\mu_3,\ldots) = \sum_{j=1}^{\infty} jH_v(\mu_j,\mu_{j+1}). \tag{3.3}$$

The CTM Hamiltonian diverges unless $\mu_j = i + 1 - j \pmod n$ for $j \gg 0$ and a certain $0 \leqslant i \leqslant n-1$.

Let $\mathcal{H}^{(i)}$ be the $\mathbb{C}$-vector space spanned by the half-infinite pure tensor vectors of the forms[1]

$$\varepsilon_{\mu_1} \otimes \varepsilon_{\mu_2} \otimes \varepsilon_{\mu_2} \otimes \cdots \quad \text{with} \quad \mu_j \in \mathbb{Z}/n\mathbb{Z}, \mu_j = i + 1 - j \pmod n \quad \text{for} \quad j \gg 0. \tag{3.4}$$

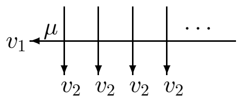
Let $\mathcal{H}^{*(i)}$ be the dual of $\mathcal{H}^{(i)}$ spanned by the half-infinite pure tensor vectors of the forms

$$\cdots \otimes \varepsilon_{\mu_{-2}} \otimes \varepsilon_{\mu_{-1}} \otimes \varepsilon_{\mu_0} \quad \text{with} \quad \mu_j \in \mathbb{Z}/n\mathbb{Z}, \quad \mu_j = i + 1 - j \pmod n \quad \text{for} \quad j \ll 0. \tag{3.5}$$

[1] We fix $\mathcal{H}^{(i)}$ by (3.4) such that it coincides with $V(\omega_i)$, the level 1 highest weight irreducible $U_q(\widehat{\mathfrak{sl}_n})$-module, in the trigonometric limit $r \to \infty$. For example, see [11], keeping in mind that our i should be read as $-i$ in [11].

Introduce the type I vertex operator by the following half-infinite transfer matrix

$$\Phi^{\mu}(v_1 - v_2) = $$



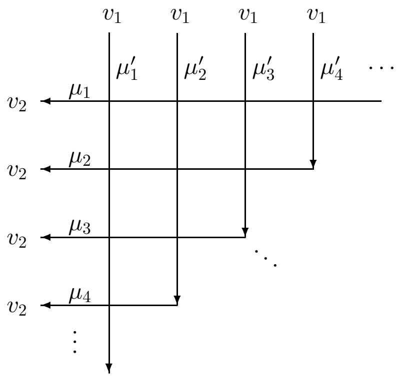
(3.6)

Then the operator (3.6) is an intertwiner from $\mathcal{H}^{(i)}$ to $\mathcal{H}^{(i+1)}$. The type I vertex operators satisfy the following commutation relation:

$$\Phi^{\mu}(v_1)\Phi^{\nu}(v_2) = \sum_{\mu',\nu'} R(v_1 - v_2)^{\mu\nu}_{\mu'\nu'} \Phi^{\nu'}(v_2)\Phi^{\mu'}(v_1). \tag{3.7}$$

Introduce the CTM in the south–east (SE) corner.

$$A^{(i)}_{SE}(v_1 - v_2)^{\mu_1\mu_2\mu_3\mu_4\cdots}_{\mu'_1\mu'_2\mu'_3\mu'_4\cdots} = $$



The diagonal form of $A^{(i)}_{\mathrm{SE}}(v)$ can be determined from the 'low-temperature' limit of the R-matrix (3.1)–(3.2):

$$A^{(i)}_{\mathrm{SE}}(v) \sim \zeta^{H_{\mathrm{CTM}}} = z^{\frac{1}{n}H_{\mathrm{CTM}}} : \mathcal{H}^{(i)} \to \mathcal{H}^{(i)}, \tag{3.8}$$

where $\sim$ refers to an equality modulo a divergent scalar in the infinite lattice limit. Likewise, other three types of the CTMs are given as follows:

$$\begin{aligned} A^{(i)}_{\mathrm{NE}}(v) : &\quad \mathcal{H}^{(i)} \to \mathcal{H}^{*(i)}, \\ A^{(i)}_{\mathrm{NW}}(v) : &\quad \mathcal{H}^{*(i)} \to \mathcal{H}^{*(i)}, \\ A^{(i)}_{\mathrm{SW}}(v) : &\quad \mathcal{H}^{*(i)} \to \mathcal{H}^{(i)}, \end{aligned} \tag{3.9}$$

where NE, NW and SW stand for the corners north–east, north–west and south–west, respectively. It seems to be rather general [3] that the product of four CTMs in the infinite lattice limit is independent of v:

$$\rho^{(i)} = A^{(i)}_{\mathrm{SE}}(v)A^{(i)}_{\mathrm{SW}}(v)A^{(i)}_{\mathrm{NW}}(v)A^{(i)}_{\mathrm{NE}}(v) = x^{2H_{\mathrm{CTM}}}. \tag{3.10}$$

Since $H(\mu_j, \mu_{j+1})$ takes on the value of $\{0, 1, \ldots, n-1\}$, the eigenvalues of H_{CTM} are of the form

$$N = \sum_{j=1}^{\infty} j m_j, \qquad 0 \leqslant m_j \leqslant n-1.$$

This stands for the partition of N such that the multiplicity of each j is at most $n-1$. Thus, the character is given by

$$\chi^{(i)} = \mathrm{tr}_{\mathcal{H}^{(i)}}(\rho^{(i)}) = \frac{(x^{2n}; x^{2n})_\infty}{(x^2; x^2)_\infty}. \tag{3.11}$$

3.2. CTM for the $A^{(1)}_{n-1}$ model

After gauge transformation [6], the CTM Hamiltonian of the $A^{(1)}_{n-1}$ model in the regime III is given as follows:

$$\begin{aligned} H_{\mathrm{CTM}}(a_0, a_1, a_2, \ldots) &= \sum_{j=1}^{\infty} j H_f(a_{j-1}, a_j, a_{j+1}), \\ H_f(a + \bar\varepsilon_\mu + \bar\varepsilon_\nu, a + \bar\varepsilon_\mu, a) &= \frac{1}{n} H_v(\nu, \mu), \end{aligned} \tag{3.12}$$

where $H_v(\nu, \mu)$ is defined by (3.2). The CTM Hamiltonian diverges unless $a_j = \xi + \omega_{i+1-j}$ for $j \gg 0$ and a certain $\xi \in P_{r-n-1}$ and $0 \leqslant i \leqslant n-1$.

For $k = a + \rho, l = \xi + \rho$ and $0 \leqslant i \leqslant n-1$, let $\mathcal{H}^{(i)}_{l,k}$ be the space of admissible paths $(a_0, a_1, a_2, \ldots)$ such that

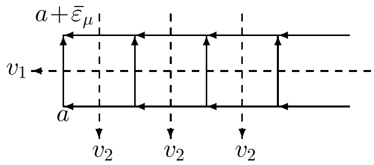
$$\begin{aligned} &a_0 = a, \quad a_j - a_{j+1} \in \{\bar\varepsilon_0, \bar\varepsilon_1, \ldots, \bar\varepsilon_{n-1}\}, \\ &\qquad \text{for} \quad j = 1, 2, 3, \ldots, \quad a_j = \xi + \omega_{i+1-j} \quad \text{for} \quad j \gg 0. \end{aligned} \tag{3.13}$$

Also, let $\mathcal{H}^{*(i)}_{l,k}$ be the space of admissible paths $(\ldots, a_{-2}, a_{-1}, a_0)$ such that

$$\begin{aligned} &a_0 = a, \quad a_j - a_{j+1} \in \{\bar\varepsilon_0, \bar\varepsilon_1, \ldots, \bar\varepsilon_{n-1}\}, \\ &\qquad \text{for} \quad j = 1, 2, 3, \ldots, \quad a_j = \xi + \omega_{i+1-j} \quad \text{for} \quad j \ll 0. \end{aligned} \tag{3.14}$$

Introduce the type I vertex operator by the following half-infinite transfer matrix

$$\Phi(v_1 - v_2)_a^{a+\bar\varepsilon_\mu} = \tag{3.15}$$

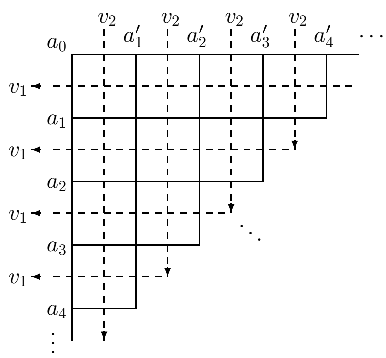


Then the operator (3.15) is an intertwiner from $\mathcal{H}^{(i)}_{l,k}$ to $\mathcal{H}^{(i+1)}_{l,k+\bar\varepsilon_\mu}$. The type I vertex operators satisfy the following commutation relation:

$$\Phi(v_1)_b^c \Phi(v_2)_a^b = \sum_d W\left[\begin{matrix} c & d \\ b & a \end{matrix}\middle| v_1 - v_2\right] \Phi(v_2)_d^c \Phi(v_1)_a^d. \tag{3.16}$$

Introduce the CTM of the $A^{(1)}_{n-1}$ model in the SE corner

$$A_{SE}^{(l,k)}(v_1 - v_2)^{a_0 a_1 a_2 a_3 a_4 \cdots}_{a_0 a'_1 a'_2 a'_3 a'_4 \cdots} =$$



The diagonal form of $A_{\mathrm{SE}}^{(l,k)}(v)$ can be determined from the 'low-temperature' limit (3.12):

$$A_{\mathrm{SE}}^{(l,k)}(v) \sim \zeta^{H_{\mathrm{CTM}}} = z^{\frac{1}{n}H_{\mathrm{CTM}}} : \mathcal{H}_{l,k}^{(i)} \to \mathcal{H}_{l,k}^{(i)}, \tag{3.17}$$

where $\sim$ refers to an equality modulo a divergent scalar in the infinite lattice limit. Likewise other three types of the CTMs are given as follows:

$$\begin{aligned} A_{\mathrm{NE}}^{(l,k)}(v) :&\quad \mathcal{H}_{l,k}^{(i)} \to \mathcal{H}_{l,k}^{*(i)},\\ A_{\mathrm{NW}}^{(l,k)}(v) :&\quad \mathcal{H}_{l,k}^{*(i)} \to \mathcal{H}_{l,k}^{*(i)},\\ A_{\mathrm{SW}}^{(l,k)}(v) :&\quad \mathcal{H}_{l,k}^{*(i)} \to \mathcal{H}_{l,k}^{(i)}. \end{aligned} \tag{3.18}$$

The product of four CTMs for the $A_{n-1}^{(1)}$ model in the infinite lattice limit is also independent of v [6]:

$$\rho_{l,k}^{(i)} = G_a x^{2nH_{l,k}^{(i)}}, \tag{3.19}$$

where

$$G_a = \prod_{\mu<\nu} [a_{\mu\nu}].$$

The character of the $A_{n-1}^{(1)}$ model was obtained in [6]:

$$\chi_{l,k}^{(i)} = \mathrm{tr}_{\mathcal{H}_{l,k}^{(i)}}\left(\rho_{l,k}^{(i)}\right) = \frac{x^{n|\beta_1 k+\beta_2 l|^2}}{(x^{2n};x^{2n})_\infty^{n-1}} G_a, \tag{3.20}$$

where

$$t^2 - \beta_0 t - 1 = (t-\beta_1)(t-\beta_2), \qquad \beta_0 = \frac{1}{\sqrt{r(r-1)}}, \qquad \beta_1 < \beta_2. \tag{3.21}$$

We note the following sum formula:

$$\sum_{\substack{k\equiv l+\omega_i \\ (\mathrm{mod}\ Q)}} \chi_{l,k}^{(i)} = \frac{(x^{2n};x^{2n})_\infty}{(x^2;x^2)_\infty} \left(\frac{(x^{2r};x^{2r})_\infty}{(x^{2r-2};x^{2r-2})_\infty}\right)^{(n-1)(n-2)/2} G'_\xi, \tag{3.22}$$

where

$$G'_\xi = \prod_{\mu<\nu} [\xi_{\mu\nu}]'.$$

Equations (3.22) and (3.11) imply that

$$\chi^{(i)} = \frac{1}{b_l} \sum_{\substack{k\equiv l+\omega_i \\ (\mathrm{mod}\ Q)}} \chi_{l,k}^{(i)}, \tag{3.23}$$

where

$$b_l = \left(\frac{(x^{2r};x^{2r})_\infty}{(x^{2r-2};x^{2r-2})_\infty}\right)^{(n-1)(n-2)/2} G'_\xi. \tag{3.24}$$

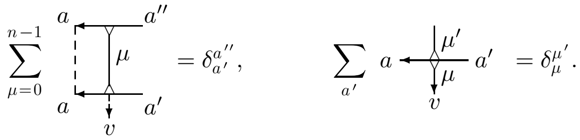


Figure 2. Picture representation of the dual intertwining vectors.

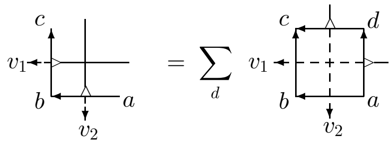


Figure 3. Vertex–face correspondence by dual intertwining vectors.

3.3. Tail operator

Let us introduce the dual intertwining vectors (see figure 2) satisfying

$$\sum_{\mu=0}^{n-1} t^*_\mu(v)^{a'}_a t^\mu(v)^a_{a''} = \delta^{a'}_{a''}, \qquad \sum_{\nu=0}^{n-1} t^\mu(v)^a_{a-\bar{\varepsilon}_\nu} t^*_{\mu'}(v)^{a-\bar{\varepsilon}_\nu}_a = \delta^\mu_{\mu'}. \tag{3.25}$$

From (2.17) and (3.25), we have (cf. figure 3)

$$t^*(v_1)^b_c \otimes t^*(v_2)^a_b R(v_1 - v_2) = \sum_d W\left[\begin{matrix} c & d \\ b & a \end{matrix}\middle| v_1 - v_2\right] t^*(v_1)^a_d \otimes t^*(v_2)^d_c. \tag{3.26}$$

Now introduce the intertwining operators between $\mathcal{H}^{(i)}$ and $\mathcal{H}^{(i)}_{l,k}$ ($k = l + \omega_i \pmod{Q}$):

$$\begin{aligned} T(u)^{\xi a_0} &= \prod_{j=0}^{\infty} t^{\mu_j}(-u)^{a_j}_{a_{j+1}} : \mathcal{H}^{(i)} \to \mathcal{H}^{(i)}_{l,k}, \\ T(u)_{\xi a_0} &= \prod_{j=0}^{\infty} t^*_{\mu_j}(-u)^{a_{j+1}}_{a_j} : \mathcal{H}^{(i)}_{l,k} \to \mathcal{H}^{(i)}, \end{aligned} \tag{3.27}$$

where $k = a_0 + \rho$ and $l = \xi + \rho$, and $0 < \Re(u) < \frac{n}{2} + 1$. The tail operator Λ (see figure 4) is defined by

$$\Lambda(u)^{a'}_a = T(u)^{\xi a'} T(u)_{\xi a}. \tag{3.28}$$

Let

$$L\left[\begin{matrix} a'_0 & a'_1 \\ a_0 & a_1 \end{matrix}\middle| u\right] := \sum_{\mu=0}^{n-1} t^*_\mu(-u)^{a_1}_{a_0} t^\mu(-u)^{a'_0}_{a'_1}. \tag{3.29}$$

Then we have

$$\Lambda(u)^{a'_0}_{a_0} = \prod_{j=0}^{\infty} L\left[\begin{matrix} a'_j & a'_{j+1} \\ a_j & a_{j+1} \end{matrix}\middle| u\right]. \tag{3.30}$$

Here we note that in the 'low-temperature' limit $t^*_j(-u)^{\xi+\omega_j}_{\xi+\omega_{j+1}} t^j(-u)^{\xi+\omega_{j+1}}_{\xi+\omega_j}$ is much greater than other, $t^*_\mu(-u)^{\xi+\omega_j}_{\xi+\omega_{j+1}} t^\mu(-u)^{\xi+\omega_{j+1}}_{\xi+\omega_j}$ $(\mu \neq j)$.

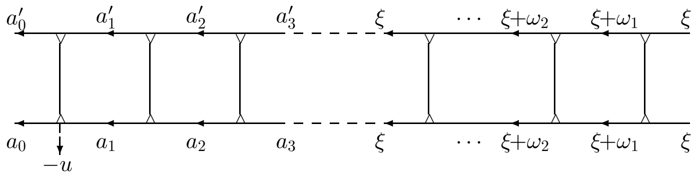
$\Lambda(u)_{a_0}^{a_0'} =$ 

Figure 4. Tail operator $\Lambda(u)_{a_0}^{a_0'}$. The upper (resp. lower) half stands for $T(u)^{\xi a_0}$ (resp. $T(u)_{\xi a_0}$).

Note that

$$L\left[\begin{matrix} a' & a'-\bar{\varepsilon}_\nu \\ a & a-\bar{\varepsilon}_\mu \end{matrix}\middle| u\right] = \frac{[u+\bar{a}_\mu - \bar{a'}_\nu]}{[u]} \prod_{j\neq\mu} \frac{[\bar{a'}_\nu - \bar{a}_j]}{[a_{\mu j}]}. \tag{3.31}$$

It is obvious from (3.25) that we have

$$L\left[\begin{matrix} a & a' \\ a & a'' \end{matrix}\middle| u\right] = \delta_{a''}^{a'}. \tag{3.32}$$

We therefore have

$$\Lambda(u)_a^a = 1. \tag{3.33}$$

From (3.23) and (3.33), we may assume that

$$\rho^{(i)} = \frac{1}{b_l} \sum_{\substack{k\equiv l+\omega_i \\ (\mathrm{mod}\ Q)}} T(u)_{\xi a} \rho_{l,k}^{(i)} T(u)^{\xi a}. \tag{3.34}$$

3.4. Commutation relations between Λ and φ

By using the vertex–face correspondence (see figure 5), we obtain

$$T(u)^{\xi b}\Phi^\mu(v) = \sum_a t^\mu(v-u)_a^b \Phi(v)_a^b T(u)^{\xi a}, \tag{3.35}$$

$$T(u)_{\xi b}\Phi(v)_a^b = \sum_\mu t_\mu^*(v-u)_b^a \Phi^\mu(v) T(u)_{\xi a}. \tag{3.36}$$

From these commutation relations and the definition of the tail operator (3.28), we have

$$\Lambda(u)_b^c \Phi(v)_a^b = \sum_d L\left[\begin{matrix} c & d \\ b & a \end{matrix}\middle| u-v\right] \Phi(v)_d^c \Lambda(u)_a^d. \tag{3.37}$$

4. The vertex operator approach

One of the most standard ways to calculate correlation functions is the vertex operator approach [10] on the basis of free field representation. In section 4.2, we recall the free field representation for the $A_{n-1}^{(1)}$ model [2]. The type I vertex operators of the $A_{n-1}^{(1)}$ model can be constructed in terms of basic bosons introduced in [12, 13]. The $A_{n-1}^{(1)}$ model has the so-called σ-invariance. The free field representation of type I vertex operator given in section 4.2 is not invariant under σ-transformation. Thus, we give other free field

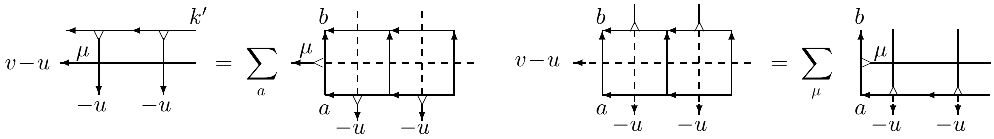


Figure 5. Commutation relations among $T(v_0)^{a\xi}$, $T(v_0)_{a\xi}$ and the type I vertex operators in vertex and face models.

representations in section 4.3. We also need the bosonized CTM Hamiltonian of the $A^{(1)}_{n-1}$ model [14] in order to obtain correlation functions of the $A^{(1)}_{n-1}$ model. In section 4.4 we discuss the space of states of the unrestricted $A^{(1)}_{n-1}$ model. The free field representation of the tail operator is presented in section 4.5.

4.1. Bosons

Let us consider the bosons, $B^j_m (1 \leqslant j \leqslant n-1, m \in \mathbb{Z}\backslash\{0\})$, with the commutation relations

$$[B^j_m, B^k_{m'}] = \begin{cases} m\dfrac{[(n-1)m]_x}{[nm]_x}\dfrac{[(r-1)m]_x}{[rm]_x}\delta_{m+m',0}, & (j=k) \\ -mx^{\mathrm{sgn}(j-k)nm}\dfrac{[m]_x}{[nm]_x}\dfrac{[(r-1)m]_x}{[rm]_x}\delta_{m+m',0}, & (j\neq k), \end{cases} \tag{4.1}$$

where the symbol $[a]_x$ stands for $(x^a - x^{-a})/(x - x^{-1})$. Define B^n_m by

$$\sum_{j=1}^{n} x^{-2jm} B^j_m = 0.$$

Then the commutation relations (4.1) holds for all $1 \leqslant j, k \leqslant n$. These oscillators were introduced in [12, 13].

For $\alpha, \beta \in \mathfrak{h}^* := \mathbb{C}\omega_0 \oplus \mathbb{C}\omega_1 \oplus \cdots \mathbb{C}\omega_{n-1}$, let us define the zero-mode operators P_α, Q_β with the commutation relations

$$[P_\alpha, \sqrt{-1}Q_\beta] = \langle\alpha, \beta\rangle, \qquad \left[P_\alpha, B^j_m\right] = \left[Q_\beta, B^j_m\right] = 0.$$

We will deal with the bosonic Fock spaces $\mathcal{F}_{l,k}$, $(l, k \in \mathfrak{h}^*)$ generated by $B^j_{-m} (m > 0)$ over the vacuum vectors $|l, k\rangle$:

$$\mathcal{F}_{l,k} = \mathbb{C}\left[\left\{B^j_{-1}, B^j_{-2}, \ldots\right\}_{1\leqslant j\leqslant n}\right]|l, k\rangle,$$

where

$$B^j_m|l, k\rangle = 0\,(m > 0),$$
$$P_\alpha|l, k\rangle = \langle\alpha, \beta_1 k + \beta_2 l\rangle|l, k\rangle,$$
$$|l, k\rangle = \exp(\sqrt{-1}(\beta_1 Q_k + \beta_2 Q_l))|0, 0\rangle,$$

where β_1 and β_2 are defined by (3.21).

4.2. Type I vertex operators

Let us define the basic operators for $j = 1, \ldots, n-1$:

$$U_{-\alpha_j}(v) = z^{\frac{r-1}{r}} : \exp\left(-\beta_1(\sqrt{-1}Q_{\alpha_j} + P_{\alpha_j}\log z) + \sum_{m\neq 0}\frac{1}{m}\left(B^j_m - B^{j+1}_m\right)(x^j z)^{-m}\right) :, \tag{4.2}$$

$$U_{\omega_j}(v) = z^{\frac{r-1}{2r}\frac{j(n-j)}{n}} : \exp\left(\beta_1(\sqrt{-1}Q_{\omega_j} + P_{\omega_j}\log z) - \sum_{m\neq 0}\frac{1}{m}\sum_{k=1}^{j} x^{(j-2k+1)m} B_m^k z^{-m}\right) :, \tag{4.3}$$

where $\beta_1 = -\sqrt{\frac{r-1}{r}}$ and $z = x^{2v}$ as usual. Following commutation relations are useful:

$$U_{\omega_1}(v)U_{\omega_j}(v') = r_j(v-v')U_{\omega_j}(v')U_{\omega_1}(v), \tag{4.4}$$

$$U_{-\alpha_j}(v)U_{\omega_j}(v') = -f(v-v',0)U_{\omega_j}(v')U_{-\alpha_j}(v), \tag{4.5}$$

$$U_{-\alpha_j}(v)U_{-\alpha_{j+1}}(v') = -f(v-v',0)U_{-\alpha_{j+1}}(v')U_{-\alpha_j}(v), \tag{4.6}$$

$$U_{-\alpha_j}(v)U_{-\alpha_j}(v') = g(v-v')U_{-\alpha_j}(v')U_{-\alpha_j}(v). \tag{4.7}$$

In the sequel we set

$$\pi_\mu = \sqrt{r(r-1)}P_{\bar{\varepsilon}_\mu}, \qquad \pi_{\mu\nu} = \pi_\mu - \pi_\nu.$$

The $\pi_{\mu\nu}$ acts on $\mathcal{F}_{l,k}$ as a scalar $\langle \varepsilon_\mu - \varepsilon_\nu, rl - (r-1)k\rangle$.

For $0 \leqslant \mu \leqslant n-1$ define the type I vertex operator [2] by

$$\begin{aligned}\phi_\mu(v) &= \oint \prod_{j=1}^{\mu} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{\omega_1}(v)U_{-\alpha_1}(v_1)\cdots U_{-\alpha_\mu}(v_\mu)\prod_{j=0}^{\mu-1} f(v_{j+1}-v_j,\pi_{j\mu})\prod_{\substack{j=0\\ j\neq\mu}}^{n-1}[\pi_{j\mu}]^{-1}\\ &= (-1)^\mu \oint \prod_{j=1}^{\mu}\frac{dz_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_\mu}(v_\mu)\cdots U_{-\alpha_1}(v_1)U_{\omega_1}(v)\\ &\quad\times \prod_{j=0}^{\mu-1} f(v_j - v_{j+1}, 1-\pi_{j\mu})\prod_{\substack{j=0\\ j\neq\mu}}^{n-1}[\pi_{j\mu}]^{-1},\end{aligned} \tag{4.8}$$

where $v_0 = v$ and $z_j = x^{2v_j}$. The integral contour for z_j-integration encircles the poles at $z_j = x^{1+2kr}z_{j-1} (k \in \mathbb{Z}_{\geqslant 0})$, but not the poles at $z_j = x^{-1-2kr}z_{j-1} (k \in \mathbb{Z}_{\geqslant 0})$, for $1 \leqslant j \leqslant \mu$.

Note that

$$\phi_\mu(v) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l,k+\bar{\varepsilon}_\mu}. \tag{4.9}$$

These type I vertex operators satisfy the following commutation relations on $\mathcal{F}_{l,k}$:

$$\phi_{\mu_1}(v_1)\phi_{\mu_2}(v_2) = \sum_{\varepsilon_{\mu_1}+\varepsilon_{\mu_2}=\varepsilon_{\mu_1'}+\varepsilon_{\mu_2'}} W\left[\begin{matrix} a+\bar{\varepsilon}_{\mu_1}+\bar{\varepsilon}_{\mu_2} & a+\bar{\varepsilon}_{\mu_1'}\\ a+\bar{\varepsilon}_{\mu_2} & a\end{matrix}\middle| v_1-v_2\right]\phi_{\mu_2'}(v_2)\phi_{\mu_1'}(v_1). \tag{4.10}$$

We thus denote the operator $\phi_\mu(v)$ by $\Phi(v)_a^{a+\bar{\varepsilon}_\mu}$ on the bosonic Fock space $\mathcal{F}_{l,a+\rho}$. We notice that our vertex operator (4.8) has different normalization from that originally constructed in [2] because of the difference of the Boltzmann weight W. Furthermore, the range of μ is shifted from that of [2] by 1 so that our $\phi_\mu(v)$ corresponds to $\phi_{\mu+1}(v)$ in [2], up to normalization.

Dual vertex operators are likewise defined as follows:

$$\phi_\mu^*(v) = c_n^{-1}\oint \prod_{j=\mu+1}^{n-1}\frac{dz_j}{2\pi\sqrt{-1}z_j} U_{\omega_{n-1}}\left(v-\frac{n}{2}\right)U_{-\alpha_{n-1}}(v_{n-1})\cdots U_{-\alpha_{\mu+1}}(v_{\mu+1})$$

$$\times \prod_{j=\mu+1}^{n-1} f(v_j - v_{j+1}, \pi_{\mu j})$$

$$= c_n^{-1}(-1)^{n-1-\mu} \oint \prod_{j=\mu+1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_{\mu+1}}(v_{\mu+1}) \cdots U_{-\alpha_{n-1}}(v_{n-1}) U_{\omega_{n-1}}\left(v - \frac{n}{2}\right)$$

$$\times \prod_{j=\mu+1}^{n-1} f(v_{j+1} - v_j, 1 - \pi_{\mu j}) \qquad (4.11)$$

where $v_n = v - \frac{n}{2}$, and

$$c_n = x^{\frac{r-1}{r}\frac{n-1}{2n}} \frac{g_{n-1}(x^n)}{(x^2; x^{2r})_\infty^n (x^{2r}, x^{2r})_\infty^{2n-3}}.$$

The integral contour for z_j-integration encircles the poles at $z_j = x^{1+2kr} z_{j+1} (k \in \mathbb{Z}_{\geqslant 0})$, but not the poles at $z_j = x^{-1-2kr} z_{j+1} (k \in \mathbb{Z}_{\geqslant 0})$, for $\mu + 1 \leqslant j \leqslant n - 1$. Note that

$$\phi_\mu^*(v) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l,k-\bar{\varepsilon}_\mu}. \qquad (4.12)$$

The operators $\phi_\mu(v)$ and $\phi_\mu^*(v)$ are dual in the following sense:

$$\sum_{\mu=0}^{n-1} \phi_\mu^*(v)\phi_\mu(v) = 1. \qquad (4.13)$$

We notice that our dual vertex operator $\phi_\mu^*(v)$ coincides with $\bar{\phi}_{\mu+1}^{*(n-1)}\left(v - \frac{n}{2}\right)$ in [2].

4.3. Other representations

The present face model has the so-called σ-invariance:

$$W\left[\begin{matrix} \sigma(c) & \sigma(d) \\ \sigma(b) & \sigma(a) \end{matrix}\middle| v\right] = W\left[\begin{matrix} c & d \\ b & a \end{matrix}\middle| v\right], \qquad \sigma(\omega_\mu) = \omega_{\mu+1}.$$

The free field representation (4.8) is not invariant under σ-transformation, so that we have other free field representations:

$$\phi_{i+\mu}(v) = \oint \prod_{j=1}^{\mu} \frac{\mathrm{d}z_j}{2\pi\sqrt{-1}z_j} U_{\omega_1}(v) U_{-\alpha_1}(v_1) \cdots U_{-\alpha_\mu}(v_\mu)$$

$$\times \prod_{j=0}^{\mu-1} f(v_{j+1} - v_j, \pi_{i+ji+\mu}) \prod_{\substack{j=0 \\ j\neq\mu}}^{n-1} [\pi_{i+ji+\mu}]^{-1}$$

$$= (-1)^\mu \oint \prod_{j=1}^{\mu} \frac{\mathrm{d}z_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_\mu}(v_\mu) \cdots U_{-\alpha_1}(v_1) U_{\omega_1}(v)$$

$$\times \prod_{j=0}^{\mu-1} f(v_j - v_{j+1}, 1 - \pi_{i+ji+\mu}) \prod_{\substack{j=0 \\ j\neq\mu}}^{n-1} [\pi_{i+ji+\mu}]^{-1}, \qquad (4.14)$$

where $v_0 = v$ and $z_j = x^{2v_j}$, and the integral contours are the same one as (4.8). In this representation the space of states $\mathcal{H}_{l,k}^{(i)}$ should be identified with $\mathcal{F}_{\sigma^{-i}(l),\sigma^{-i}(k)}$.

4.4. Free field realization of CTM Hamiltonian

Let

$$H_F = \sum_{m=1}^{\infty} \frac{[rm]_x}{[(r-1)m]_x} \sum_{j=1}^{n-1} \sum_{k=1}^{j} x^{(2k-2j-1)m} B_{-m}^{k} \left(B_m^j - B_m^{j+1}\right) + \frac{1}{2} \sum_{j=1}^{n-1} P_{\omega_j} P_{\alpha_j}$$
$$= \sum_{m=1}^{\infty} \frac{[rm]_x}{[(r-1)m]_x} \sum_{j=1}^{n-1} \sum_{k=1}^{j} x^{(2j-2k-1)m} \left(B_{-m}^j - B_{-m}^{j+1}\right) B_m^k + \frac{1}{2} \sum_{j=1}^{n-1} P_{\omega_j} P_{\alpha_j} \tag{4.15}$$

be the CTM Hamiltonian on the Fock space $\mathcal{F}_{l,k}$ [14]. Then we have the homogeneity relation

$$\phi_\mu(z) q^{H_F} = q^{H_F} \phi_\mu(q^{-1} z) \tag{4.16}$$

and

$$\mathrm{tr}_{\mathcal{F}_{l,k}}(x^{2nH_F} G_a) = \frac{x^{n|\beta_1 k + \beta_2 l|^2}}{(x^{2n}; x^{2n})_\infty^{n-1}} G_a. \tag{4.17}$$

By comparing (3.20) and (4.17), we conclude that $\rho_{l,k}^{(i)} = G_a x^{2nH_F}$ and $\mathcal{H}_{l,k}^{(i)} = \mathcal{F}_{l,k}$, where $k = a + \rho$.

The relation between $\rho^{(i)}$ and $\rho_{l,k}^{(i)}$ is as follows:

$$\rho^{(i)} = \sum_{\substack{k \equiv l + \omega_i \\ (\mathrm{mod}\ Q)}} T(u)_{\xi a} \frac{\rho_{l,k}^{(i)}}{b_l} T(u)^{\xi a}. \tag{4.18}$$

4.5. Free field realization of tail operators

Consider (3.37) for $(c, b, a) \to (a, a + \bar{\varepsilon}_0 + \bar{\varepsilon}_\mu, a - \bar{\varepsilon}_\mu)$, where $\mu \neq 0$. The coefficient L diverges when $u \to v$, so that we obtain the following necessary condition:

$$\prod_{\substack{j=1 \\ j \neq \mu}}^{n-1} [a_{0j}] \Phi(v)_{a-\bar{\varepsilon}_0}^{a} \Lambda(v)_{a-\bar{\varepsilon}_\mu}^{a-\bar{\varepsilon}_0} + \prod_{\substack{j=1 \\ j \neq \mu}}^{n-1} [a_{\mu j}] \Phi(v)_{a-\bar{\varepsilon}_\mu}^{a} \Lambda(v)_{a-\bar{\varepsilon}_\mu}^{a-\bar{\varepsilon}_\mu} = 0. \tag{4.19}$$

By solving (4.19), we obtain

$$\Lambda(u)_{a-\bar{\varepsilon}_\mu}^{a-\bar{\varepsilon}_0} = G_\pi \oint \prod_{j=1}^{\mu} \frac{\mathrm{d}z_j}{2\pi\sqrt{-1} z_j} U_{-\alpha_1}(v_1) \cdots U_{-\alpha_\mu}(v_\mu) \prod_{j=0}^{\mu-1} f(v_{j+1} - v_j, \pi_{j\mu}) G_\pi^{-1}, \tag{4.20}$$

where

$$G_\pi := \prod_{\kappa < \lambda} [\pi_{\kappa\lambda}].$$

Note that a free field representation of $\Lambda(u)_{a-\bar{\varepsilon}_\mu}^{a-\bar{\varepsilon}_\nu}$ for $\nu > 0$ can be constructed on $\mathcal{F}_{\sigma^{-\nu}(l), \sigma^{-\nu}(k)}$.

In the following section, we need a tail operator $\Lambda(u)_{a-\sum_{j=1}^N \bar{\varepsilon}_{\mu_j}}^{a-\sum_{j=1}^N \bar{\varepsilon}_{\nu_j}}$ in order to calculate n-point functions. This type tail operator can be represented in terms of free bosons. In order to show this fact, let us introduce the symbol $\lesssim$ as follows. We say $\mu \lesssim \nu$ if $0 \leqq \mu_0 \leqq \nu_0 \leqq n-1$ and $\mu = \mu_0$ (mod n), $\nu = \nu_0$ (mod n).

It is clear that there exists $0 \leqq i \leqq n-1$ such that

$$\sharp\{j | \nu_j + i \lesssim 0\} > 0,$$

and

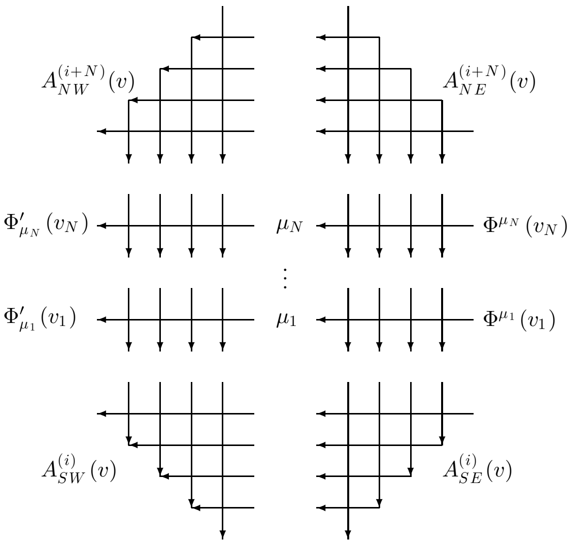
$$\sharp\{j|\mu_j+i\lesssim m\}\leqq\sharp\{j|\nu_j+i\lesssim m\},$$

for every $0\leqq m\leqq n-1$. In this case a free field representation of the tail operator $\Lambda(u)^{a-\sum_{j=1}^N\bar{\varepsilon}_{\nu_j}}_{a-\sum_{j=1}^N\bar{\varepsilon}_{\mu_j}}$ can be constructed on $\mathcal{F}_{\sigma^{-i}(l),\sigma^{-i}(k)}$.

5. Correlation functions

5.1. General formulae

Consider the local state probability (LSP) such that the state variable at jth site is equal to μ_j $(1\leqq j\leqq N)$, under a certain fixed boundary condition. In order to obtain LSP, it is convenient to divide the lattice into four transfer matrices and $2N$ vertex operators as follows:



Here, the incoming vertex operator $\Phi'_\mu(v)$ should be distinguished form the outgoing vertex operator $\Phi^\mu(v)$.

Let us consider the normalized partition function with fixed $\mu_1,\ldots,\mu_N$:

$$\begin{aligned}P^{(i)}_{\mu_1\ldots\mu_N}(v_1,\ldots,v_N):=&\frac{1}{\chi^{(i)}}\mathrm{tr}_{\mathcal{H}^{(i)}}\big(A^{(i)}_{\mathrm{SW}}(v)\Phi'_{\mu_1}(v_1)\cdots\Phi'_{\mu_N}(v_N)\\&\times A^{(i+N)}_{\mathrm{NW}}(v)A^{(i+N)}_{\mathrm{NE}}(v)\Phi^{\mu_N}(v_N)\cdots\Phi^{\mu_1}(v_1)A^{(i)}_{\mathrm{SE}}(v)\big).\end{aligned}\tag{5.1}$$

In the vertex operator approach [10], the LSP can be given by $P^{(i)}_{\mu_1\ldots\mu_N}(v_1,\ldots,v_N)|_{v_1=\cdots=v_N=v}$. In what follows, we denote $P^{(i)}_{\mu_1\ldots\mu_N}=P^{(i)}_{\mu_1\ldots\mu_N}(v_1,\ldots,v_N)|_{v_1=\cdots=v_N=v}$.

The YBE and the crossing symmetry imply the following relation [8]:

$$\Phi'_\mu(v')A^{(i+1)}_{\mathrm{NW}}(v)A^{(i+1)}_{\mathrm{NE}}(v)=A^{(i)}_{\mathrm{NW}}(v)A^{(i)}_{\mathrm{NE}}(v)\Phi^*_\mu(v').\tag{5.2}$$

Thus, one-point local state probability of the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model can be given by

$$\begin{aligned}P^{(i)}_j&=\frac{1}{\chi^{(i)}}\mathrm{tr}_{\mathcal{H}^{(i)}}\left(\Phi^*_j(v)\Phi^j(v)\rho^{(i)}\right)\\&=\frac{1}{\chi^{(i)}}\sum_{\substack{k\equiv l+\omega_i\\(\mathrm{mod}\ Q)}}\mathrm{tr}_{\mathcal{H}^{(i)}_{l,k}}\left(T(u)^{\xi a}\Phi^*_j(v)\Phi^j(v)T(u)_{\xi a}\frac{\rho^{(i)}_{l,k}}{b_l}\right)\end{aligned}$$

$$= \frac{1}{\chi^{(i)}} \sum_{\substack{k\equiv l+\omega_i \\ (\mathrm{mod}\ Q)}} \sum_{\mu,\nu} t_j^*(v-u)^a_{a+\bar\varepsilon_\nu} t^j(v-u)^{a+\bar\varepsilon_\nu}_{a+\bar\varepsilon_\nu-\bar\varepsilon_\mu}$$

$$\times \mathrm{tr}_{\mathcal{H}^{(i)}_{l,k}} \left(\Phi^*(v)^a_{a+\bar\varepsilon_\nu} \Phi(v)^{a+\bar\varepsilon_\nu}_{a+\bar\varepsilon_\nu-\bar\varepsilon_\mu} \Lambda(u)^{a+\bar\varepsilon_\nu-\bar\varepsilon_\mu}_a \frac{\rho^{(i)}_{l,k}}{b_l} \right). \qquad (5.3)$$

Here in the third equality of (5.3), we use (3.35) and the fact that $\Phi^*_j(v)$, $\Phi^*(v)^a_{a+\bar\varepsilon_\nu}$ and $t^*_j(v)^a_{a+\bar\varepsilon_\nu}$ are given by the fusion of $n-1\Phi^k(v)$'s, $\Phi(v)^{a+\bar\varepsilon_\mu}_a$'s and $t^k(v)^{a+\bar\varepsilon_\mu}_a$'s, respectively.

In general, N-point local state possibility of this model can be given by

$$P^{(i)}_{j_N\ldots j_1} = \frac{1}{\chi^{(i)}} \mathrm{tr}_{\mathcal{H}^{(i)}} \left(\Phi^*_{j_N}(v)\cdots\Phi^*_{j_1}(v)\Phi^{j_1}(v)\cdots\Phi^{j_N}(v)\rho^{(i)} \right)$$

$$= \frac{1}{\chi^{(i)}} \sum_{\substack{k\equiv l+\omega_i \\ (\mathrm{mod}\ Q)}} \sum_{\substack{a_1\ldots a_N \\ a'_1\ldots a'_N}} t^*_{j_N}(v-u)^a_{a_N}\cdots t^*_{j_1}(v-u)^{a_2}_{a_1} t^{j_1}(v-u)^{a_1}_{a'_1}\cdots t^{j_N}(v-u)^{a'_{N-1}}_{a'_N}$$

$$\times \mathrm{tr}_{\mathcal{H}^{(i)}_{l,k}} \left(\Phi^*(v)^a_{a_N}\cdots\Phi^*(v)^{a_2}_{a_1}\Phi(v)^{a_1}_{a'_1}\cdots\Phi(v)^{a'_{N-1}}_{a'_N}\Lambda(u)^{a'_N}_a \frac{\rho^{(i)}_{l,k}}{b_l} \right), \qquad (5.4)$$

where the second sum on the second line should be taken such that $(a, a_N), \ldots, (a_2, a_1)$ and $(a'_1, a_1), \ldots, (a'_N, a'_{N-1})$ are all admissible.

5.2. Spontaneous polarization

In this section, we reproduce the expression for spontaneous polarization [8]:

$$\langle g\rangle^{(i)} = \sum_{j=0}^{n-1} \omega^j P^{(i)}_j = \omega^{i+1} \frac{(x^2;x^2)^2_\infty}{(x^{2r};x^{2r})^2_\infty} \frac{(\omega x^{2r};x^{2r})_\infty(\omega^{-1}x^{2r};x^{2r})_\infty}{(\omega x^2;x^2)_\infty(\omega^{-1}x^2;x^2)_\infty}, \qquad (5.5)$$

by performing traces and n-fold integrals on (5.3). In [8], expression (5.5) was obtained by solving a system of difference equations, the quantum Knizhnik–Zamolodchikov equations of level $-2n$.

First we replace $a+\bar\varepsilon_\nu$ by a for simplicity:

$$P^{(i)}_j = \frac{1}{\chi^{(i)}} \sum_{\substack{k\equiv l+\omega_{i+1} \\ (\mathrm{mod}\ Q)}} \sum_{\mu,\nu} t^*_j(v-u)^{a-\bar\varepsilon_\nu}_a t^j(v-u)^a_{a-\bar\varepsilon_\mu}$$

$$\times \mathrm{tr}_{\mathcal{H}^{(i+1)}_{l,k}} \left(\Phi(v)^a_{a-\bar\varepsilon_\mu} \Lambda(u)^{a-\bar\varepsilon_\mu}_{a-\bar\varepsilon_\nu} \frac{\rho^{(i)}_{l,k-\bar\varepsilon_\nu}}{b_l} \Phi^*(v)^{a-\bar\varepsilon_\nu}_a \right). \qquad (5.6)$$

We note that

$$\sum_{j=0}^{n-1} \omega^j t^*_j(v-u)^{a-\bar\varepsilon_\nu}_a t^j(v-u)^a_{a-\bar\varepsilon_\mu} = \frac{[v-u+a_{\mu\nu}]_\omega}{[v-u]} \prod_{\substack{j=0 \\ j\neq\nu}}^{n-1} \frac{[a_{\mu j}]_\omega}{[a_{\nu j}]}, \qquad (5.7)$$

where

$$[v]_\omega = x^{\frac{v^2}{r}-v}\Theta_{x^{2r}}(\omega x^{2v}).$$

Thus, the spontaneous polarization can be reduced as

$$\langle g\rangle^{(i)} = \frac{1}{\chi^{(i)}} \sum_{\mu=0}^{n-1} \langle g\rangle^{(i)}_\mu,$$

where

$$\langle g\rangle_\mu^{(i)} = \sum_{\substack{k\equiv l+\omega_{i+1}\\ (\mathrm{mod}\ Q)}} \sum_{\nu=0}^{n-1} \frac{[v-u+a_{\mu\nu}]_\omega}{[v-u]} \prod_{\substack{j=0\\ j\neq\nu}}^{n-1} \frac{[a_{\mu j}]_\omega}{[a_{\nu j}]} \mathrm{tr}_{\mathcal{H}_{l,k}^{(i+1)}} \\ \times \left(\Phi(v)_{a-\bar{\varepsilon}_\mu}^{a} \Lambda(u)_{a-\bar{\varepsilon}_\nu}^{a-\bar{\varepsilon}_\mu} G_{a-\bar{\varepsilon}_\nu} \Phi^*(v+n)_a^{a-\bar{\varepsilon}_\nu} \frac{x^{2nH_{l,k}^{(i+1)}}}{b_l} \right).$$

When $\mu = 0$, in order to calculate the operator product $\Phi(v)_{a-\bar{\varepsilon}_0}^{a} \Lambda(u)_{a-\bar{\varepsilon}_\nu}^{a-\bar{\varepsilon}_0} G_{a-\bar{\varepsilon}_\nu}$ $\Phi^*(v+n)_a^{a-\bar{\varepsilon}_\nu}$, the following operator product formulae are useful:

$$c_n^{-1} U_{\omega_1}(v) U_{-\alpha_1}(v_1) \cdots U_{-\alpha_{n-1}}(v_{n-1}) U_{\omega_{n-1}}\left(v+\frac{n}{2}\right) \\ = x^{-\frac{n-1}{2}\frac{r-1}{r}} (x^2;x^{2r})_\infty^n (x^{2r};x^{2r})_\infty^{2n-3} z^{-\frac{r-1}{nr}} \prod_{j=0}^{n-1} z_j^{-\frac{r-1}{r}} \frac{(x^{2r-1}z_{j+1}/z_j;x^{2r})_\infty}{(xz_{j+1}/z_j;x^{2r})_\infty} \\ \times :U_{\omega_1}(v) U_{-\alpha_1}(v_1) \cdots U_{-\alpha_{n-1}}(v_{n-1}) U_{\omega_{n-1}}\left(v+\frac{n}{2}\right):, \tag{5.8}$$

where $z_0 = z$, and $z_n = x^n z$. Using (5.8) we have the following trace formulae:

$$\mathrm{tr}_{\mathcal{H}_{l,k}^{(i+1)}} \left(c_n^{-1} U_{\omega_1}(v) U_{-\alpha_1}(v_1) \cdots U_{-\alpha_{n-1}}(v_{n-1}) U_{\omega_{n-1}}\left(v+\frac{n}{2}\right) x^{2nH_{l,k}^{(i+1)}} \right) \\ = x^{n\left(\frac{r-1}{r}|k^2|-2\langle k,l\rangle+\frac{r}{r-1}|l|^2\right)} x^{2\frac{r-1}{r}\sum_{j=1}^{n-1} a_{0j}\left(v_{j+1}-v_j-\frac{1}{2}\right)-2\sum_{j=1}^{n-1}\xi_{0j}\left(v_{j+1}-v_j-\frac{1}{2}\right)} \\ \times (x^{2n};x^{2n})_\infty (x^{2r};x^{2r})_\infty^{2n-3} \frac{(x^2;x^{2n},x^{2r})_\infty^n}{(x^{2r+2n-2};x^{2n},x^{2r})_\infty^n} \\ \times \prod_{j=0}^{n-1} \frac{(x^{2r-1}z_{j+1}/z_j;x^{2n},x^{2r})_\infty (x^{2r+2n-1}z_j/z_{j+1};x^{2n},x^{2r})_\infty}{(xz_{j+1}/z_j;x^{2n},x^{2r})_\infty (x^{2n+1}z_j/z_{j+1};x^{2n},x^{2r})_\infty}. \tag{5.9}$$

Let us denote the rhs of (5.9) by $A_{l,k}^{(i)}(v;v_1,\ldots,v_{n-1})$. Then we have

$$\langle g\rangle_0^{(i)} = \frac{1}{b_l} \sum_{\substack{k\equiv l+\omega_{i+1}\\ (\mathrm{mod}\ Q)}} \prod_{0<j<k} [a_{jk}] \sum_{\nu=0}^{n-1} \frac{[v-u+a_{0\nu}]_\omega}{[v-u]} \oint_{C_\nu} \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\sqrt{-1}} \\ \times \prod_{\substack{j=0\\ j\neq\nu}}^{n-1} \frac{[a_{0j}]_\omega}{[a_{\nu j}]} f(v_{j+1}-v_j, 1-a_{\nu j}) A_{l,k}^{(i)}(v;v_1,\ldots,v_{n-1}). \tag{5.10}$$

Here, the integral contour C_ν is chosen such that

$$|z_j| = \begin{cases} x^j(|z|+j\varepsilon) & (1 \leqq j \leqq \nu) \\ x^j(|z|-(n-j)\varepsilon) & (\nu+1 \leqq j \leqq n-1), \end{cases}$$

where $\varepsilon > 0$ is a very small positive number.

Let us denote the rhs of (5.10) by $H_l^{(i)}$. As noted in the previous section, the trace on $\mathcal{H}_{l,k}^{(i)}$ should be taken on $\mathcal{F}_{\sigma^{-\mu}(l),\sigma^{-\mu}(k)}^{(i)}$ for $\mu > 0$. Thus, $\langle g\rangle_\mu^{(i)}$ can be reduced to $H_{\sigma^{-\mu}(l)}^{(i)}$. Let

$$B_{l,k}^{(i)}(v,u) := x^{n\left(\frac{r-1}{r}|k^2|-2\langle k,l\rangle+\frac{r}{r-1}|l|^2\right)} x^{2a_{0n-1}(v-u)} \tilde{G}_a, \qquad \tilde{G}_a = \prod_{j=1}^{n-1} [a_{0j}]_\omega \prod_{0<j<k} [a_{jk}].$$

Consider the following sum,

$$S^{(i)}(v,u) := \frac{[0]_\omega}{[v-u]} \sum_{\mu=0}^{n-1} \sum_{\substack{k\equiv l+\omega_{i+1}\\ (\mathrm{mod}\ Q)}} B_{\sigma^{-\mu}(l),\sigma^{-\mu}(k)}^{(i)}(v,u),$$

and take the limit $u \to v$[2]. Then we have

$$\lim_{u\to v} S^{(i)}(v,u) = \omega^{i+1} b_l \frac{(\omega x^{2r};x^{2r})_\infty(\omega^{-1}x^{2r};x^{2r})_\infty}{(x^{2r};x^{2r})_\infty^2} \frac{(x^2;x^2)_\infty(x^{2n};x^{2n})_\infty^n}{(\omega;x^2)_\infty(\omega^{-1}x^2;x^2)_\infty}. \tag{5.11}$$

This can be confirmed by comparing the series expansion in x of both sides order by order.

Here we cite the sum formula from [2]:

$$\sum_{\nu=0}^{n-1}\prod_{\substack{j=0\\ j\neq\nu}}^{n-1} \frac{f(v_{j+1}-v_j, 1-\pi_{\nu j})}{[\pi_{\nu j}]} = 0. \tag{5.12}$$

This can be derived by applying Liouville's second theorem to the following elliptic function:

$$F(w) = \prod_{j=0}^{n-1} \frac{\left[v_{j+1}-v_j-\frac{1}{2}+w-\pi_j\right]}{\left[v_{j+1}-v_j-\frac{1}{2}\right][w-\pi_j]}.$$

On equation (5.10), the contour for z_1-integral is common for all ν except for $\nu = 0$. Thus, by using (5.12), $H_l^{(i)}$ can be evaluated by the residue at $z_1 = x^{1+2u} \to x^1 z$. The resulting $(n-2)$-fold integral has the same structures of both the integrand and the contour as the original $(n-1)$-fold one, except for the number of integral variables by one. Thus, we can repeat this evaluation procedure $n-1$ times to find

$$H_l^{(i)} \sim \frac{1}{[v-u]}\frac{1}{b_l}\frac{B_{l,k}^{(i)}}{(x^{2n};x^{2n})_\infty^{n-1}}, \tag{5.13}$$

at $u \sim v$. Substituting (5.11) and (5.13) into

$$\langle g\rangle^{(i)} = \frac{1}{\chi^{(i)}}\sum_{\mu=0}^{n-1} H^{(i)}_{\sigma^{-\mu}(l)}, \tag{5.14}$$

we reproduce the expression for the spontaneous polarization (5.5) originally obtained in [8].

6. Concluding remarks

In this paper, we constructed a free field representation method in order to obtain correlation functions of Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model. The essential point was to find a free field representation of the tail operator $\Lambda_a^{a'}$, the nonlocal operator which intertwines the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model and $A_{n-1}^{(1)}$ model. As a consistency check, we perform $(n-1)$-fold integrals and traces for the one-point function to reproduce the expression of the spontaneous polarization originally obtained in [8].

There are some related works concerning the eight-vertex model and its higher spin version. A bootstrap approach for the eight-vertex model was presented in [15]. The vertex operators of the eight-vertex model with some special values of r were directly bosonized in [16]. A free field representation method for form factors of the eight-vertex model was constructed in [17]. A higher spin generalization of the free field representation method was achieved in [18]. As for the $(\mathbb{Z}/n\mathbb{Z})$-symmetric model, it is important to consider the extension to the form factor problem or the application to the fused model. We wish to address these problems in future.

[2] Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model does not have the parameter u so that all the physical quantities should be independent of u. Thus, we set $u \to v$ here, in order to avoid some difficulty.

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